Analysis of optical directional couplers using shortcuts to adiabaticity

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Abstract: In this paper, we propose the use of the invariant based shortcuts to adiabaticity for the analysis of directional couplers. By describing the dynamical evolution of the system using the eigenstates of the invariant through new parameterizations, the system stability against errors in coupling coefficient and propagation constants mismatch is connected with the new parameters, which can be linked back to system parameters through inverse engineering. The merits and limitations of the conventional tapered directional coupler designs with various window functions are obtained through the analysis. We then propose an optimal design of compact directional couplers that is stable against errors in input wavelength and coupling coefficient simultaneously. The designed directional coupler has better tolerance, as compared to the conventional resonant couplers with smooth shape functions of Hamming and Blackman. These results are verified by beam propagation simulations.

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References and links
1. **Introduction**

Directional couplers with tolerance to variations in fabrication and wavelength are highly desirable with many interesting and significant applications in integrated optics and optical communications [1–5]. Conventionally, the analysis of directional couplers are based on the solution of coupled mode equations [6], where the spectral properties and fabrication tolerance can be solved for directional couplers having a constant spacing and a uniform cross section. On the other hand, tapered directional couplers with changes in the device geometry can be designed to suppress the side lobes in the device spectra, and coupled-mode theory analysis of such devices leads to a nonlinear differential equation that can only be solved numerically, shedding little insights on the device characteristics beyond some special cases [7–9].

Recently, a family of quantum control protocols called “shortcuts to adiabaticity (STA)” [10] has been exploited in light manipulation in optical waveguide structures [11–18]. For instance, the invariant-based inverse engineering [19] and counter-diabatic driving (or quantum transitionless algorithm) [20, 21] have been directly applied to obtain fast and robust mode conversion/splitting in multimode waveguides [11, 12]. As a matter of fact, invariant-based inverse engineering and quantum transitionless algorithm are mathematically equivalent [21], therefore both approaches are capable of realizing the shortcuts for directional couplers in optical waveguides [13–16]. These applications of various STA protocols to the design of coupled-waveguide devices illustrate the analogies one can draw between quantum mechanics and wave optics [22], and that the robustness of STA protocols can be directly translated to these waveguide devices. In fact, these STA protocols, while originally developed to accelerate “slow” adiabatic passages in various quantum systems, also shed lights on the system dynamics beyond the adiabatic limits. In particular, the inverse engineering, combined with perturbation theory and optimal control, provides a versatile toolbox for designing the optimal shortcuts with respect to different error and perturbation [23–25], in terms of invariant dynamics.

In this paper, we present a systematic STA analysis of directional couplers using the inverse engineering based on the Lewis-Riesenfeld invariants. Combined with the perturbation theory, the STA based analysis allows one to handle directional couplers with spatially varying geometry based on the invariant dynamics. Tapered designs using the Hamming and Blackman window functions [7, 8] are analyzed with the new approach, showing the merits and limitations...
of these designs. Finally, we use the invariant based STA to minimize the error sensitivities of directional couplers to wavelength and fabrication errors simultaneously by optimizing the dynamical invariant. The spatially varying coupling coefficient and propagation constant mismatch are inversely engineered to achieve a compact and robust directional coupler. The performance of various directional couplers are compared to demonstrate that the proposed directional coupler has better tolerance, as compared to the common parallel coupler and the directional couplers with various window functions.

2. The STA description of power coupling in directional couplers

2.1. The invariant based STA

In this section, we introduce the inverse engineering based on the Lewis-Risenfeld invariant and its connection to power evolution in directional couplers. Consider a directional coupler of length $L$, consisting of two waveguides placed in proximity with propagation constants $\beta_+(z)$ and $\beta_-(z)$, see Fig. 1, where the cross-section of the waveguide structure is shown. Under the scalar and paraxial approximation and assuming weak coupling, the changes in the guided-mode amplitudes in the individual waveguides $\Psi = [A_+, A_-]^T$ with propagation distance is described by coupled-mode equations as, $i d\Psi/dz = H_0(z)\Psi$, that is

$$i \frac{d}{dz} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix},$$ \hspace{1cm} (1)

where $\Omega \equiv \Omega(z)$ (real) is the coupling coefficient, and $\Delta \equiv \Delta(z) = [\beta_-(z) - \beta_+(z)]$ describes propagation constants mismatch between the waveguides. Replacing the spatial variation $z$ with the temporal variation $t$, Eq. (1) is equivalent to the time-dependent Schrödinger equation ($\hbar \equiv 1$) describing the interaction dynamics of a two-state system driven by a coherent laser excitation [22], in which $\Omega$ and $\Delta$ are Rabi frequency and detuning, respectively. In optical waveguides, the propagation constant mismatch $\Delta$ is linearly dependent on the width difference, and the coupling coefficient $\Omega$ can be related to the separation between two waveguides [26]. The conventional coupled mode theory analysis describes the evolution of $\Psi$ using the guided-modes in the individual waveguides as the basis, and the mode coupling behavior is obtained by solving Eq. (1).

The idea of the invariant based STA is that one can describe the evolution of $\Psi$ using the eigenstates of the dynamical invariant $I(z)$ with the invariant satisfying $\partial_z I + (1/i)[I, H_0] = 0$ [27]. And the eigenstates of the invariant are decoupled during system evolution. We can parameterize the eigenstates of the invariant as

$$|\psi_0(z)\rangle = e^{i\gamma/2} \begin{bmatrix} \cos \frac{\gamma}{2} e^{-i\beta/2} \\ \sin \frac{\gamma}{2} e^{i\beta/2} \end{bmatrix},$$ \hspace{1cm} (2)
and the orthogonal one (for all times \( \langle \psi_0(z) | \psi_\perp(z) \rangle = 0 \))

\[
|\psi_\perp(z)\rangle = e^{-i\gamma z/2} \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\beta z/2} \\ -\cos \frac{\theta}{2} e^{i\beta z/2} \end{pmatrix}.
\]

(3)

The system evolution is now described by these new parameters. To construct the Hamiltonian (obtain the device parameters \( \Omega \) and \( \Delta \)) inversely, we substitute Eqs. (2) or (3) directly into the Schrödinger equation Eq. (1), and obtain the following auxiliary differential equations:

\[
\dot{\theta} = -\Omega \sin \beta,
\]

(4)

\[
\dot{\beta} = -\Omega \cot \theta \cos \beta - \Delta,
\]

(5)

\[
\dot{\gamma} = \dot{\theta} \cot \beta/ \sin \theta.
\]

(6)

These equations are equivalent to those obtained by the invariant dynamical theory [27], since \(|\psi_0(z)\rangle \langle \psi_0(z)| (|\psi_\perp(z)\rangle \langle \psi_\perp(z)|)\) is a dynamical invariant [23–25].

To describe power coupling between two waveguides along \(|\psi_0(z)\rangle\), up to a phase factor, from the initial states \(|\psi_0(0)\rangle = |2\rangle \equiv (\rangle_0\rangle)\) to the final state \(|\psi_0(L)\rangle = |1\rangle \equiv (\rangle_1\rangle)\), we should set the following boundary conditions at \(z = 0\) and \(z = L\):

\[
\theta(0) = \pi, \quad \theta(L) = 0.
\]

(7)

Based on Eqs. (4)–(6), we shall design the coupling coefficient, \( \Omega \), and propagation constant mismatch, \( \Delta \), by inverting Eqs. (4)–(6). In principle, we can choose arbitrary ansatz to interpolate the function of \( \theta \), satisfying the boundary conditions, to achieve power coupling. The freedom allows us to combine the inverse engineering and optimal control to select the most robust state dynamics in presence of various noise and systematic errors [23–25].

### 2.2. Perturbation theory

In optical waveguides, there exist both input wavelength and coupling coefficient errors from fabrication at the same time, which can be described by the systematic errors in detuning and Rabi frequency, namely, \( \mathbf{H}' = (\hbar/2)(\eta_\Omega \mathbf{\Omega} \mathbf{\sigma}_x + \eta_\Delta \mathbf{\sigma}_z) \), with Pauli matrix \( \mathbf{\sigma}_{x,z} \) and the error amplitude \( \eta_{\Omega,\Delta} \).

In the error-free case, we consider the family of protocols that results in perfect state transfer from \( |2\rangle \) to \( |1\rangle \): the unperturbed solution, \( |\Psi(z)\rangle = |\psi_0(z)\rangle \), satisfying Eq. (7). By assuming the ideal, unperturbed Hamiltonian, \( \mathbf{H}_0 \), given by Eq. (1), we consider \( \mathbf{H}' \) as the perturbation in our perturbation theory calculation. Finally, the efficiency of the system to be in state \( |1\rangle \) at \( z = L \) (coupling efficiency) is calculated as, \( P = \langle 1|\Psi(L)\rangle|^2 \approx 1 - \frac{1}{4} \int_0^L dz \langle \psi_\perp(z)|\mathbf{H}'|\psi_0(z)\rangle|^2 \) [16,23], which turns out to be

\[
P \approx 1 - \frac{1}{4} \left( \int_0^L dz e^{i\gamma z} (\eta_\Delta \sin \theta + 2\eta_\Omega \dot{\theta} \sin^2 \theta) \right)^2.
\]

(8)

For further analysis, we rewrite the approximate coupling efficiency as \( P \approx 1 - \eta_\Delta^2 q_\Delta - \eta_\Omega^2 q_\Omega \), where \( q_{\Delta,\Omega} \) are the error sensitivities, defined as

\[
q_\Omega = \frac{1}{2} \frac{\partial^2 P_1}{\partial \eta_\Omega^2} = \frac{1}{4} \left( \int_0^L dz e^{i\gamma z} 2\dot{\theta} \sin^2 \theta \right)^2,
\]

(9)

\[
q_\Delta = \frac{1}{2} \frac{\partial^2 P_1}{\partial \eta_\Delta^2} = \frac{1}{4} \left( \int_0^L dz e^{i\gamma z} \sin \theta \right)^2.
\]

(10)

It is clear that the coupling efficiencies as a function of errors in \( \Omega \) and \( \Delta \) are now related to our new parameterizations introduced in Eqs. (2) and (3). By nullifying these two quantities...
Fig. 2. Parameters for directional coupler design, including waveguide width $W_{L,R}$ and separation $D$, for (a) resonant protocols and (b) optimal protocol.

above and inversely obtain the Hamiltonian through the auxiliary equations Eqs. (4)-(6), one can obtain robust directional coupler designs with respect to wavelength and coupling coefficient errors simultaneously. On the other hand, one could also contemplate designing sensitive filters by maximizing these integrals.

3. Applications to device analysis

3.1. Resonant coupler ($\Delta = 0$) with uniform spacing ($\Omega = \text{constant}$)

First of all, we shall discuss the simplest case of directional coupler with no propagation constants mismatch and a uniform spacing. With $\Delta = 0$, we obtain $\beta = \pi/2$ from Eq. (5). To obtain a constant $\Omega$, the function of $\theta$ is chosen as

$$\theta_l(z) = -\pi(s - 1),$$

with $s = z/L$, which results in $\Omega = \pi/L$. Clearly, such coupler works like the flat $\pi$ pulse in quantum optics, which is the shortest design but sensitive to the parameter variations.
3.2. Resonant coupler ($\Delta = 0$) with tapered coupling ($\Omega(z)$)

To suppress side lobes in the simple directional couplers, several window functions, such as Hamming and Blackman [7, 8], are used in directional couplers with tapered coupling coefficients. With the right boundary conditions, the Hamming and Blackman functions of $\theta(z)$ are given by

$$\theta_h(z) = 0.426 \sin (2\pi s) - \pi (s-1),$$

$$\theta_b(z) = 0.5952 \sin (2\pi s) - 0.0476 \sin (4\pi s) - \pi (s-1).$$

(12)

(13)

The STA analysis can give further insights into the error stability of these approaches that are not previously available using the coupled mode analysis. We describe the results as follows.

When considering solely the coupling coefficient variations, described by the error in Rabi frequency, we have the efficiency for the resonant coupling ($\Delta = 0$) as follows [23],

$$P = \frac{1}{2} - \frac{1}{2} \cos \left[ (1 + \eta_\Omega) \int_0^L |\Omega| dz \right].$$

(14)

This tells us that no matter how we design the function of $\theta$, the stability with respect to the coupling efficient error can’t be improved for resonant coupler ($\beta = \pi/2$), since the “pulse area”

$$\int_0^L |\Omega| dz = \int_0^L |\theta| dz = \pi,$$

(15)

needs to be fulfilled. In this case, Eq. (9) gives $q_\Omega = \pi^2/4$, and the efficiency is calculated as $P \approx 1 - \pi^2 \eta_\Omega^2 / 4$, which is consistent with Eq. (14), when $\eta_\Omega$ is perturbative.

Regarding the input wavelength fluctuation, we consider it as the systematic error in detuning through the propagation constants mismatch. So the efficiency can be obtained by $P \approx 1 - \eta_\Delta q_\Delta$, where Eq. (10) gives

$$q_\Delta = \frac{1}{4} \left[ \int_0^L dz \sin \theta \right]^2,$$

(16)

with $\beta = \pi/2$ and $\gamma = 0$ for resonant couplers. In principle, $q_\Delta$ can be nullified by a sudden jump from $\theta(0) = 0$ to $\theta(L) = \pi$. This, however, results in infinite coupling coefficient $\Omega$ as shown by Eq. (4). So, smooth window functions such as the Hamming and the Blackman functions can improve the stability with respect to input wavelength variations, since the value of the integral in Eq. (16) is reduced as compared to using the linear function in Eq. (11). The STA analysis clearly shows that the window functions for $\theta$, see Eqs. (12) and (13), only improve the stability against input wavelength error but not coupling coefficient error.

3.3. Design of an optimal coupler

<table>
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<th>Table 1. The parameters $c_1$ and $c_2$ for optimization.</th>
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<td>Linear</td>
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<td>Blackman</td>
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<td>Hamming</td>
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In what follows, we shall design a directional coupler with higher stability with respect to two errors simultaneously using the STA. We shall design the function $\theta$ and $\beta$ to nullify both $q_\Omega$
and $q_\Delta$, which guarantees the best tolerance with respect to the variation of input wavelength and coupling coefficient. For simplicity, we choose $\theta_l(z)$ without loss of generality, and impose [25]

$$
\gamma(z) = \theta + c_1 \sin(2\theta) + c_2 \sin(4\theta),
$$

(17)

with two free parameters $c_1$ and $c_2$. Numerical calculation gives $c_1 = -1.12$ and $c_2 = 0.84$, resulting in $q_\Delta = 0$ and $q_{\Omega} = 0$, and the value of following integral is

$$
\int_0^L |\Omega| dz = \int_0^L |-\dot{\theta}/ \sin \beta| dz = 2.31\pi,
$$

(18)

which is larger than $\pi$, see Eq. (15). Once the parameters $c_1$ and $c_2$ are fixed, one can solve for $\beta$ by Eq. (6). Actually, one can also use the Hamming and Blackman functions of $\theta$ to achieve the $c_1$ and $c_2$ for optimization, see Table 1. Finally, the detuning and Rabi frequency can be inversely designed by Eqs. (4)–(6), and the optimal directional coupler is thus achieved. To clarify how it works, we choose the simpler case of linear function $\theta_l(z)$, as an example in the following beam propagation simulations.
Fig. 4. Coupling efficiency versus (a) input wavelength, (b) spacing, and (c) waveguide width errors.
4. BPM simulation and discussion

Next, we turn to perform the numerical simulation for polymer waveguides using beam propagation method (BPM) [28]. The cross-section of the polymer channel waveguide structure is shown in Fig. 1, where the waveguide spacing $D$ and widths $W_{L,R}$ are adjusted along the propagation direction to satisfy the designed set of $\Omega$ and $\Delta$. In detail, the relation between $\Delta$ and the width difference $\delta W = W_L - W_R$ can be approximated by a linear relation [1], and the relation between $\Omega$ and waveguide separation $D$ in a symmetric coupler is well fitted by the exponential relation $\Omega = \Omega_0 \exp[-\kappa(D - D_0)]$ [26]. As a consequence, the waveguide parameters can be achieved by such mapping. The length of the devices is set at $L = 2000 \mu m$, and the central wavelength is set at $1.55 \mu m$. For the resonant couplers, the waveguide width is $W_L = W_R = 2 \mu m$ and $\delta W = 0$. Also the waveguide separation $D$ is obtained from $\Omega$ by using different $\theta$, see Fig. 2(a). For the optimal design, we obtain the parameters, $D$ and $W_{L,R}$ by using the inverse engineering, as shown in Fig. 2(b).

Figure 3 displays the BPM simulations for resonant couplers with fixed and tapered coupling and the optimal coupler, where the geometry of the designed directional couplers are shown. The power is completely coupled from one waveguide to the other. The coupling efficiencies versus different errors are shown in Fig. 4. The designed optimal directional coupler has better tolerance, as compared to the resonant couplers. Furthermore, the numerical simulation verifies the analysis on the use of window functions in section 3.2. In Fig. 4(a), using different $\theta$ functions actually improves the stability, which agrees with the theoretical analysis, also see Eq. (16). And in Fig. 4(b), it is clear that the stability doesn’t change with the chosen window function. Remarkably, for the optimal directional coupler, the bandwidth is increased from 1.35 $\mu m$ to 1.85 $\mu m$, for coupling efficiencies greater than 95%. The spacing error window is also increased, with greater than 95% coupling efficiency from $-0.80 \mu m$ to $0.50 \mu m$. These results show significant improvement over conventional designs with a minimized spacing error and a flat wavelength window. The performance of the devices subject to variations in the waveguide widths is also investigated in Fig. 4(c). Width variations lead to errors in both $\Omega$ and $\Delta$ simultaneously in the optimal coupler. For the optimal coupler, the fabrication tolerance for greater than 95% coupling efficiency is from $+0.40 \mu m$ down to $-1.0 \mu m$. For the resonant couplers, the fabrication tolerance in width variation is from around $-0.30 \mu m$ to $0.60 \mu m$, with better tolerance on the positive variation side, because $\Omega$ is increased due to larger modal overlap. The insertion loss of the optimal coupler is $0.2 dB$ larger than that of the resonant couplers due to the sharp corners seen in Fig. 3(a). These sharp turns also limit the minimum device length achievable with the STA technique, as a shorter device length would lead to sharper turns in the resulting coupler.

5. Conclusion

In conclusion, we have systematically introduced the STA as an new way for the analysis of directional couplers. New insights on the merits and limitations of the conventional tapered directional coupler designs with various window functions are obtained through the analysis. It is shown that the use of window functions only improves the device stability against wavelength errors. We also applied the STA to design a compact and optimal directional couplers against spacing and wavelength errors simultaneously. BPM simulations demonstrate that the designed directional coupler is compact with better robustness, as compared to the conventional couplers, even with smooth window functions of Hamming and Blackman. The STA technique can be applied in general to weakly coupled waveguide systems described by the coupled mode theory. For applications to high index contrast waveguides such as silicon photonic waveguides [29], care must be taken to ensure that the device is working in the weak coupling regime.
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