From Eq. (10.7), the two pitch radii are given by

\[
    r_{p_2} = \frac{N_2}{2P_d} = \frac{20}{2(4)} = 2.5 \text{ in}
\]

and

\[
    r_{p_1} = \frac{N_3}{2P_d} = \frac{30}{2(4)} = 3.75 \text{ in}
\]

The length of the line of contact is given by Eq. (10.17) as

\[
    \lambda = -r_{p_2} \sin \phi + \sqrt{a_z^2 + 2a_r r_{p_2} + r_{p_2}^2 \sin^2 \phi - r_{p_3} \sin \phi + \sqrt{a_z^2 + 2a_r r_{p_3} + r_{p_3}^2 \sin^2 \phi}}
\]

\[
    = -2.5 \sin 20^\circ + \sqrt{0.25^2 + 2(0.25)(2.5) + 2.5^2 \sin^2 20^\circ}
\]

\[
    -3.75 \sin 20^\circ + \sqrt{0.25^2 + 2(0.25)(3.75) + 3.75^2 \sin^2 20^\circ}
\]

\[
    = 1.185 \text{ in}
\]

From Eq. (10.18), the contact ratio is

\[
    m_c = \frac{\lambda}{p_b} = \frac{1.185}{0.7380} = 1.6052
\]

Therefore, on average, approximately 1.6 teeth are in contact as the gears mesh.

10.7 INVOLUTOMETRY

It is important, for the purpose of analyzing stress and deflection of gear teeth, to be able to compute the thickness of a tooth at any radius. From Fig. 10.15, the arc distance \( AB \) is equal to the linear distance \( BT \). Therefore,

\[
    AB = BT = r_b \tan \xi
\]

and

\[
    \chi = \frac{AB}{r_b} - \xi = \tan \xi - \xi
\]

where \( \chi \) is called the involute function of \( \xi \) and is written

\[
    \text{inv}(\xi) = \tan \xi - \xi
\]

Involute functions can be obtained from tables. They are also easily calculated on a calculator. The thickness of the tooth at any radius \( r \) can be computed using involute functions.

**FIGURE 10.15** The basic geometry for the definition of the involute function.
Referring to Fig. 10.16, we have

\[ \frac{t_p}{2r_p} = \alpha - \text{inv}\phi \]

\[ \frac{t}{2r} = \alpha - \text{inv}\beta \]

Hence

\[ \frac{t}{2r} = \frac{t_p}{2r_p} + \text{inv}\phi - \text{inv}\beta \]  \hspace{1cm} (10.20)

where

\[ \cos\beta = \frac{r_h}{r} = \frac{r_p \cos\phi}{r} \]  \hspace{1cm} (10.21)

**FIGURE 10.16** Computation of the tooth thickness, \( t \), at any radius, \( r \). Tooth thickness is computed as a curvilinear length along the circumference of the circle of radius \( r \) through point \( T \).

**EXAMPLE 10.2**

*Thickenss of a Gear Tooth*

**Solution**

Find the thickness at the addendum and base circles of a tooth of diametral pitch 4 on a gear with 30 teeth cut with a 20° pressure angle, standard dimensions, and with the tooth thickness at the pitch circle equal to half the circular pitch.

First compute the radii at the three locations for which the tooth thickness is desired. The radius of the pitch circle is given by Eq. (10.7) as

\[ r_p = \frac{N}{2P_d} = \frac{30}{2(4)} = 3.75 \text{ in} \]
Using Table 10.1 gives the addendum radius of

\[ r_a = r_p + a = r_p + \frac{1}{P_d} = 3.75 + \frac{1}{4} = 4.0 \text{ in} \]

The base circle radius is given by Eq. (10.5) as

\[ r_b = r_p \cos \phi = 3.75 \cos 20^\circ = 3.524 \text{ in} \]

From Table 10.1, the circular tooth thickness is

\[ t_p = \frac{\pi}{2P_d} = \frac{\pi}{2(4)} = 0.393 \text{ in} \]

Next compute the angle \( \beta \) at the addendum circle. From Eq. (10.21), we have

\[ \beta_a = \cos^{-1} \left( \frac{r_p \cos \phi}{r_a} \right) = \cos^{-1} \left( \frac{3.75 \cos 20^\circ}{4.0} \right) = 28.241^\circ \]

The value of \( \beta \) at the base circle is zero. Therefore, the tooth thickness at the base circle is

\[ t_b = 2r_b \left( \frac{t_p}{2r_p} + \text{inv} \phi - \text{inv} \beta_a \right) = 2(3.524) \left( \frac{0.393}{2(3.75)} + \text{inv}20^\circ - \text{inv}28.241^\circ \right) = 0.474 \text{ in} \]

From Example 10.2, it is clear that the gear tooth thickness becomes smaller as the radius increases from the base circle radius. In the limiting case, the tooth thickness is zero. For the gear in Example 10.2, determine the maximum radius that could be specified for the addendum circle if a nonstandard gear were used.

From Eq. (10.20) and Fig. 10.17, the gear tooth thickness becomes zero when

\[ 2r \left( \frac{t_p}{2r_p} + \text{inv} \phi - \text{inv} \beta \right) = 0 \]  \hspace{1cm} (10.22)

Equation (10.22) is satisfied when

\[ \text{inv} \beta = \frac{t_p}{2r_p} + \text{inv} \phi = \frac{0.393}{2(3.75)} + \text{inv}20^\circ = 0.0673 \]

This equation can be solved for \( \beta \) by using a table of values for the involute function or by simply computing a series of values for \( \text{inv} \beta \) on a calculator or with a program such as MATLAB. The solution is \( \beta = 32.13^\circ \). From Fig. 10.17, it is clear that \( \beta \) and \( r \) are related by

\[ r = r_b / \cos \beta \]  \hspace{1cm} (10.23)

Therefore,

\[ r = r_b / \cos \beta = 3.524 / \cos 32.13^\circ = 4.161 \text{ in} \]  \hspace{1cm} (10.24)
10.8 INTERNAL GEARS

An internal or annular gear is a gear that has its center on the same side of the pitch circle as the pinion meshing with it, as shown in Fig. 10.18. The addendum circle for the internal gear is inside the pitch circle, and the teeth are concave rather than convex. Internal gears are commonly used in planetary gear systems and compact gear boxes. The primary advantage of an internal gear set is the compactness of the drive. Also, both the pinion and gear rotate in the same direction. Other advantages are the lower contact stresses because the surfaces conform better than external gear sets. There are also lower relative sliding between teeth and a greater length of contact possible between mating teeth since there is no limit to the involute profile on the flank of the internal gear.

FIGURE 10.17 Limiting case for \( r \) (tooth thickness = 0).

FIGURE 10.18 Internal gear and pinion.
Because of the tooth shape, the bending strength of the internal teeth is much greater than the strength of the teeth on the pinion. Therefore, the pinion is always the weaker member unless different materials are used for the two gears.

As in the case of external gears, the contact occurs along the line of action that is tangent to both base circles and passes through the pitch point or point of tangency between the two pitch circles. Referring to Fig. 10.18, we see that the line of action is tangent to the base circle of the internal gear at $C$ and to the base circle of the pinion at $B$. If contact occurs between $B$ and $C$, interference will result because the involute part of the pinion tooth does not cross the line of action until $B$ is reached. Contact continues until point $A$ is reached, where $A$ is the intersection of the line of action with the addendum of the pinion. Therefore, the length of the line of action is $AB$, and the addendum of the gear should not extend beyond point $B$.

A different type of interference between the gear and pinion can also occur when the number of teeth on both gears varies only slightly. The interference is called fouling, and it occurs at inactive profiles when the teeth of the pinion withdraw from the space of the gear. Potential sites for fouling are locations $a$ and $b$ in Fig. 10.18. To remove the potential for fouling, internal gears are usually cut with a shaper cutter that has two teeth fewer than the internal gear being cut. This automatically relieves the tips of the internal gear and eliminates the potential for fouling for any pinion with fewer teeth than the cutter.$^3$

Standard tooth proportions are not used for internal gears because the addendum of the gears must be shorter than those given in Table 10.1 to avoid interference. However, the basic equations for circular and diametral pitch apply, and the contact ratio and angles of action can be determined in the same manner as for external gears.

### 10.9 GEAR MANUFACTURING

Gear teeth can be formed in a variety of ways including various forms of casting: sand casting, investment casting, and die casting. They can be cut from flat stock using electron-discharge machining (EDM), CNC milling, and even precision sawing (and secondary machining or grinding). Gears made from polymers, aluminum, magnesium, and so on can be extruded and cut to width. Thin gears can be blanked from sheet stock.

Gears that must carry large loads relative to their overall size are usually made of high-strength materials such as steel. Some gears, such as bevel gears, can be forged; however, the vast majority of production gears are machined from blanks. Very small quantities can be machined using EDM, CNC milling, or horizontal milling with a formed cutter; however, when large volumes are involved, the machining is usually done with a generating cutter. In formed cutters, the tooth takes the exact shape of the cutter. Therefore, a separate cutter is technically required for each gear pitch and each number of teeth because the shape of the space between the teeth varies with both pitch and tooth number. In reality, the change in space is not significant in many cases. For a given pitch, only eight cutters are required to cut any gear in the range of 12 teeth to a rack with reasonable accuracy. However, such gears are usually not accurate enough for high speeds. A separate set of cutters is required for each pitch.$^4$ Figure 10.19 shows an example of milling using a formed cutter.

In a generating cutter, the tool has a shape different from the tooth profile, and the tool is moved relative to the gear blank to produce the desired gear shape.
When large volumes are involved, the fabrication of gears normally involves the following steps:

1. Blank fabrication,
2. Forming of teeth,
3. Refining of teeth,
4. Heat treatment,
5. Grinding, deburring, and cleaning, and
6. Finish coating.

Blank fabrication involves all of the general and special features of the gear blank. This includes forming the hub and keyways. Tooth generation includes machining the gear teeth using one of the processes discussed later in this section. The refining operations include shaving, grinding, burnishing, and lapping and are used to improve the accuracy of the gear teeth. These are necessary to remove machining marks and to improve the gear quality. The higher the gear quality, the greater the power rating and the lower the noise generated by meshing gear pairs. Heat treatment includes case hardening, which is necessary for high-performance gears to improve the resistance to surface pitting and tooth fracture. If heat treated, the gears must be reground if high accuracy is required. Deburring and cleaning are essential for all gears regardless of how they have been manufactured or the accuracy desired. Finish coatings include processes such as anodizing aluminum or depositing diamond films, but they may involve only grease or paint. The objective is to improve corrosion resistance, to reduce friction and wear, or simply to improve appearance.
Machining by shaping and hobbing are the most common methods of gear tooth generation. The objective is to slowly mesh a gear blank into a cutting tool that has teeth that will be conjugate with the teeth cut into the blank. The cutting action is always orthogonal to the side of the gear blank. In the shaping process, the cutter looks like a gear but is made of much harder material (see Figs. 10.20 and 10.21). When the gear is shaped, the reciprocating shaper cutter is moved radially into the blank until the pitch circle of the cutter and the gear blank are tangent. After each cutting stroke, the cutter is raised above the blank, and both the blank and the cutter rotate a very small amount on their pitch circles. The process continues until all of the teeth are cut. Shaping is the main method used to produce internal gears and gears integral with a shaft that has a shoulder next to the gear.

The shaping cutter can also be in the form of a rack. In this case, the pitch circle of the cutter is a straight line, and the cutter moves on a straight line tangent to the pitch circle of the gear. Relative to the gear, the pitch line of the rack appears to roll around the pitch circle of the gear. The gear teeth are formed by the envelope of the rack teeth, as shown in Fig. 10.22.

![FIGURE 10.20 Forming gear teeth with shaper.](image)

![FIGURE 10.21 Shaping an internal gear. (Courtesy of Fellows Corporation, Springfield, Vermont.)](image)
Hobbing is a method of generating gear teeth that is geometrically similar to generating the teeth with a rack cutter. The teeth of a hob are of the same shape as those on a rack cutter. However, the teeth are attached to a helical path on a cylindrical cutter. The hob looks like a worm gear with horizontal slices taken out of it. It is similar in appearance to a machine screw tap. The hob action is shown in Fig. 10.23, and a hobbing machine is shown.
in Fig. 10.24. In the hobbing machine, the hob teeth are aligned with the axis of the gear teeth. The hob and blank are rotated continuously at the proper angular velocity ratio, and the hob is fed slowly across the face of the width of the blank from one side to the other to form the teeth. The hobbing process is the most popular among the machining processes because it produces the most teeth in a given time and because the same tool can be used to cut helical as well as spur gear teeth. However, for internal gears, a gear shaper must be used.

FIGURE 10.24 Hobbing spur gear teeth. (Courtesy of Bourn & Koch.)

10.10 INTERFERENCE AND UNDERCUTTING

If the path of contact extends beyond the point of tangency with the base circle of the pinion, called the interference point, the tips of the teeth on the gear come into contact with portions of the pinion tooth profile inside the base circle. Because the involute is not defined inside the base circle, conjugate action is lost, and, in fact, the tips of the gear teeth will interfere with the lower portion of the pinion tooth flank if the tooth is machined with a formed cutter. If the pinion is generated with a rack-type cutter or hob, the cutter teeth will undercut the pinion teeth and weaken them. This situation is shown in Fig. 10.25. Thus, it is very undesirable to have the path of contact extend past the interference point.

In Fig. 10.26, two gears are shown in mesh. The interference point on gear 2 is point $A$ and that on gear 3 is point $D$. The addendum circle of gear 3 intersects the line of action at $B$, and the addendum circle of gear 2 intersects the line of action at $C$. If the addendum circle for gear 2 extends beyond point $D$ or the addendum circle of gear 3 extends beyond point $A$, interference will occur. Actually, we need only investigate one of these conditions because, as indicated in Fig. 10.26, interference will always occur first on the smaller gear. Therefore, we need only check point $D$. In Fig. 10.26, if we visualize the pitch diameter of gear 2 becoming larger, point $C$ will gradually move toward point $D$. Eventually, there will be a diameter that causes point $C$ to move beyond point $D$, and interference will occur. If gear 2 is actually a shaper cutter, the material in the interference region will simply be
removed. If, however, the teeth on gear 3 were formed by a smaller shaper cutter or cut directly, for example, by using a conforming milling cutter, then material might exist in the problem region. In that case, when gear 2 is meshed with gear 3, there will be volumetric interference, and the gear loads could be high enough to cause premature failure.

The worst case for interference or undercutting is when the pinion is meshed with a rack or generated with a rack cutter. This condition is shown in Fig. 10.27. Therefore, if we design the gear to avoid interference with a rack, the gear will mesh with all other standard gears satisfying the criteria in Section 10.5.2. Undercutting becomes more severe as the number of teeth on the gear is reduced because, for a given diametral pitch, the pitch diam-
FIGURE 10.27 Interference will occur at all contact locations within the base circle.

eter decreases when the number of teeth is reduced. Therefore, there will be a critical number of teeth on the gear to just avoid undercutting. The minimum number of teeth to avoid undercutting can be determined by identifying when the addendum circle of the rack cutter extends beyond the interference point on the gear. This can be done using the simplified geometry in Fig. 10.28. Using that figure, we get

\[ v = \frac{a}{\sin \phi} = r_p \sin \phi \]

or

\[ r_p = \frac{a}{\sin^2 \phi} \]

(10.25)

FIGURE 10.28 Geometry for determining undercutting.
We may use \( k/P_d \) as the addendum for a standard gear, where \( k \) is a constant (1 for full-depth gears and 0.8 for stub tooth gears), and from Eq. (10.7) we have

\[
2r_p = \frac{N}{P_d}
\]

Substituting into Eq. (10.25) gives

\[
N = \frac{2k}{\sin^2 \phi}
\]  

(10.26)

Results for some of the common pressure angles and systems are summarized in Table 10.2. Note that the number of teeth must be a whole number for a continuously rotating gear. Also notice that the minimum number of teeth varies inversely with the pressure angle. Therefore, the minimum number of teeth for a \( 14^{1/2}_0 \) pressure angle is 32, whereas for a \( 20^\circ \) pressure angle it is 18. This is one of the reasons why the \( 14^{1/2}_0 \) system is rarely used in modern machinery.

**TABLE 10.2 Minimum Number of Teeth to Avoid Undercutting for Standard Gears**

<table>
<thead>
<tr>
<th>System</th>
<th>Full depth</th>
<th>Full depth</th>
<th>Full depth</th>
<th>Stub</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>( 14^{1/2}_0 )</td>
<td>( 20^\circ )</td>
<td>( 25^\circ )</td>
<td>( 20^\circ )</td>
</tr>
<tr>
<td>( k )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>( N )</td>
<td>31.9</td>
<td>17.10</td>
<td>11.20</td>
<td>13.68</td>
</tr>
<tr>
<td>( N_{\text{min}} )</td>
<td>32</td>
<td>18</td>
<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>

**10.11 NONSTANDARD GEARING**

Sometimes, for reasons of saving space, the aforementioned minima are violated. Interference can still be avoided by offsetting the cutting rack so that the addendum line of the rack passes through the interference point or outside it. This requires a larger blank for the pinion. The net effect is to increase the addendum of the pinion and decrease its dedendum. This is shown in Table 10.3. The removal of undercutting is accompanied by other beneficial effects: The tooth shape is stronger and the contact ratio is improved. However, a non-standard gear results.

Referring to Fig. 10.29, we see that the cutter can be offset a distance \( e \) to bring the addendum line through the pitch point. This offset distance can be computed from

\[
e = a + r_b \cos \phi - r_p
\]  

(10.27)

where \( \phi \) is the cutting pressure angle and \( r_p \) is the cutting pitch radius. We can eliminate \( r_b \) from Eq. (10.27) using Eq. (10.5) and the expression \( \sin^2 \phi + \cos^2 \phi = 1 \). Then,

\[
e = a + r_p \cos^2 \phi - r_p = a - r_p \sin^2 \phi
\]
**TABLE 10.3 Minimum Values of Addendum for Unequal Addendum and Dedendum Gears to Avoid Undercut**

<table>
<thead>
<tr>
<th>Number of pinion teeth</th>
<th>AGMA 20° coarse-pitch system</th>
<th>Minimum number of gear teeth</th>
<th>AGMA 20° fine-pitch system</th>
<th>Minimum number of gear teeth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a_p)</td>
<td>(a_G)</td>
<td>(a_p)</td>
<td>(a_G)</td>
</tr>
<tr>
<td>7</td>
<td>—</td>
<td>—</td>
<td>1.4143^b</td>
<td>0.4094</td>
</tr>
<tr>
<td>8</td>
<td>—</td>
<td>—</td>
<td>1.43669^b</td>
<td>0.4679</td>
</tr>
<tr>
<td>9</td>
<td>—</td>
<td>—</td>
<td>1.4190^b</td>
<td>0.5264</td>
</tr>
<tr>
<td>10</td>
<td>1.468</td>
<td>0.532</td>
<td>25</td>
<td>1.4151</td>
</tr>
<tr>
<td>11</td>
<td>1.409</td>
<td>0.591</td>
<td>24</td>
<td>1.3566</td>
</tr>
<tr>
<td>12</td>
<td>1.351</td>
<td>0.649</td>
<td>23</td>
<td>1.2982</td>
</tr>
<tr>
<td>13</td>
<td>1.292</td>
<td>0.708</td>
<td>22</td>
<td>1.2397</td>
</tr>
<tr>
<td>14</td>
<td>1.234</td>
<td>0.766</td>
<td>21</td>
<td>1.1812</td>
</tr>
<tr>
<td>15</td>
<td>1.117</td>
<td>0.825</td>
<td>20</td>
<td>1.1227</td>
</tr>
<tr>
<td>16</td>
<td>1.117</td>
<td>0.883</td>
<td>19</td>
<td>1.0642</td>
</tr>
<tr>
<td>17</td>
<td>1.058</td>
<td>0.942</td>
<td>18</td>
<td>1.0057</td>
</tr>
<tr>
<td>18</td>
<td>1.000</td>
<td>1.000</td>
<td>18</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

\(^a\)The values in this table are for one diametral pitch; divide them by the desired diametral pitch to obtain the addendum.

\(^b\)Not proportional to the increase in tooth thickness; a reasonable top land must be provided.

---

**FIGURE 10.29** Geometry for determining undercut when the hob is withdrawn by an amount \(e\).
The expression for the tooth thickness can be used to relate tooth thickness at any radius to the actual meshing pressure angle \((\phi_m)\)

\[
t_{m_2} = 2r_{m_2}\left(\frac{t_2}{2r_2} + \text{inv}\phi - \text{inv}\phi_m\right)
\]

\[
t_{m_3} = 2r_{m_3}\left(\frac{t_3}{2r_3} + \text{inv}\phi - \text{inv}\phi_m\right)
\]

where \(t_{m_2}\) and \(t_{m_3}\) are the thicknesses at the meshing pitch circle, and \(r_{m_2}\) and \(r_{m_3}\) are the meshing pitch radii.

Now

\[
t_{m_2} + t_{m_3} = \frac{2\pi r_{m_2}}{N_2}
\]

and

\[
\frac{r_{m_3}}{r_{m_2}} = \frac{r_3}{r_2} = \frac{N_3}{N_2}
\]

Therefore,

\[
t_{m_2} + t_{m_3} = 2r_{m_2}\left(\frac{t_2}{2r_2} + \text{inv}\phi - \text{inv}\phi_m\right) + 2r_{m_3}\left(\frac{t_3}{2r_3} + \text{inv}\phi - \text{inv}\phi_m\right)
\]

gives

\[
\frac{2\pi r_{m_2}}{N_2} = 2r_{m_2}\left(\frac{t_2}{2r_2} + \text{inv}\phi - \text{inv}\phi_m\right) + \frac{2N_3}{N_2}r_{m_2}\left(\frac{t_3}{2r_3} + \text{inv}\phi - \text{inv}\phi_m\right)
\]

or

\[
\frac{2\pi}{N_2} = \frac{t_2}{r_2} + \frac{2(N_2 + N_3)}{N_2}\left(\text{inv}\phi - \text{inv}\phi_m\right)
\]

giving

\[
\text{inv}\phi_m = \text{inv}\phi - \frac{2\pi r_2 - N_2(t_2 + t_3)}{2r_2(N_2 + N_3)}
\]

(10.28)

This equation permits computation of \(\phi_m\) given the cutting pitch radius, \(r_2\), of the gear, the cutting pressure angle, \(\phi\), the tooth numbers, \(N_2\) and \(N_3\), the tooth thickness, \(t_2\), at the cutting pitch circle of the gear, and the tooth thickness, \(t_3\), of the pinion at the cutting pitch radius.

Referring to Fig. 10.30, we have

\[
t_3 = t^* + 2e \tan \phi
\]

and

\[
t^* = \frac{P_e}{2} = \frac{\pi}{2P_d}
\]
Hence

\[ t_3 = \frac{\pi}{2P_d} + 2e \tan \phi \]  

(10.29)

but

\[ e = a - r_3 \sin^2 \phi \]

or

\[ e = \frac{1}{P_d} - \frac{N_3}{2P_d} \sin^2 \phi \]  

(10.30)

The equations for spur tooth gearing are summarized in Table 10.4.

### Table 10.4 Summary of Spur Gear Formulas

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diametral pitch</td>
<td>( P_d = \frac{N}{d_p} = \frac{N}{2r_p} = \frac{\pi}{p_c} )</td>
</tr>
<tr>
<td>Circular pitch</td>
<td>( p_c = \frac{\pi}{p_d} = \frac{\pi d_p}{N} = \pi m = \frac{p_b}{\cos \phi} = \frac{\pi d_b}{N \cos \phi} )</td>
</tr>
<tr>
<td>Base pitch</td>
<td>( p_b = \frac{\pi d_b}{N} = p_c \cos \phi = \frac{\pi d_p \cos \phi}{N} )</td>
</tr>
<tr>
<td>Module</td>
<td>( m = \frac{d_p}{N} = \frac{1}{P_d} = \frac{p_c}{\pi} )</td>
</tr>
<tr>
<td>Tooth thickness at pitch circle</td>
<td>( t = \frac{p_c}{2} = \frac{\pi}{2P_d} = \frac{\pi d_p}{2N} = \frac{\pi m}{2} = \frac{p_b}{2 \cos \phi} = \frac{\pi d_b}{2N \cos \phi} )</td>
</tr>
<tr>
<td>Pitch diameter</td>
<td>( d_p = \frac{N}{P_d} = \frac{N p_c}{\pi} = N m )</td>
</tr>
<tr>
<td>Outside diameter</td>
<td>( d_o = d_p + 2a ) (typically ( a = k/P_d ), where ( k = 1 ) or 0.8)</td>
</tr>
</tbody>
</table>
### TABLE 10.4 continued

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root diameter</td>
<td>( d_r = d_p - 2b )</td>
</tr>
<tr>
<td>(typically ( b = q/P_d ), where ( q = 1.25, 1.2, ) or ( 1 ))</td>
<td></td>
</tr>
<tr>
<td>Base Circle Diameter</td>
<td>( d_b = d_p \cos \phi )</td>
</tr>
<tr>
<td>Center distance</td>
<td>( C = r_{p_2} + r_{p_3} = \frac{d_{p_2} + d_{p_3}}{2} = \frac{N_2 + N_3}{2P_d} = \frac{p_2(N_2 + N_3)}{2\pi} )</td>
</tr>
<tr>
<td>Length of line of contact</td>
<td>( \lambda = -r_{p_2} \sin \phi + \frac{\sqrt{a_2^2 + 2a_2r_{p_2} + r_{p_2}^2 \sin^2 \phi}}{r_{p_2}} )</td>
</tr>
<tr>
<td></td>
<td>( -r_{p_3} \sin \phi + \frac{\sqrt{a_3^2 + 2a_3r_{p_3} + r_{p_3}^2 \sin^2 \phi}}{r_{p_3}} )</td>
</tr>
<tr>
<td>Contact ratio</td>
<td>( m_c = \frac{\lambda}{p_b} = \frac{\lambda}{p_c \cos \phi} = \frac{\lambda P_d}{\pi \cos \phi} )</td>
</tr>
<tr>
<td>Velocity ratio</td>
<td>( R = r_{p_2} = \frac{d_{p_2}}{d_{p_3}} = \frac{N_3}{N_2} )</td>
</tr>
<tr>
<td>Tooth thickness at radius ( r )</td>
<td>( t = 2r \left[ \frac{l_p}{2r_p} + \text{inv}\phi - \text{inv}\beta \right] )</td>
</tr>
<tr>
<td></td>
<td>( \cos \beta = \frac{r_b}{r} = \frac{r_p \cos \phi}{r} )</td>
</tr>
<tr>
<td>Involute function</td>
<td>( \text{inv}\phi = \tan \phi - \phi )</td>
</tr>
<tr>
<td>Number of teeth at undercutting</td>
<td>( N = \frac{2aP_d}{\sin^2 \phi} )</td>
</tr>
<tr>
<td>Hob withdrawal for no undercutting</td>
<td>( e = \frac{1}{P_d} - \frac{N_3}{2P_d} \sin^2 \phi &gt; 0.03994 )</td>
</tr>
</tbody>
</table>

---

**EXAMPLE 10.4**  
Computing Nonstandard Gear Geometry  
**Solution**

A 13-tooth pinion with diametral pitch 6 and 20° cutting pressure angle is to mate with a 50-tooth gear. Find the center distance, and the meshing pressure angle if the pinion is cut with a standard cutter offset so that the addendum line passes through the interference point. (Compare with the corresponding values for a standard pinion.)

From Eq. (10.30),

\[
e = \frac{1}{P_d} - \frac{N_3}{2P_d} \sin^2 \phi = \frac{1}{6} - \frac{13}{2 \times 6} \sin^2 20^\circ = 0.03994
\]

From (10.29),

\[
t_1 = \frac{\pi}{2P_d} + 2e \tan \phi = \frac{\pi}{2 \times 6} + 2 \times 0.03994 \tan 20^\circ = 0.29087
\]

From Eq. (10.7),

\[
r_2 = \frac{N_2}{2P_d} = \frac{50}{2 \times 6} = 4.1667
\]
and from Table 10.1, 

\[ t_2 = \frac{2 \pi r_2}{2N_s} = 0.26180 \]

From Eq. (10.28)

\[ \text{inv} \phi_m = \text{inv} 20^\circ - \frac{2 \pi \times 4.1667 - 50(0.29087 + 0.26180)}{2 \times 4.1667(50 + 13)} = 0.017673 \]

To find the value for \( \phi_m \), it is possible to use tables or the value can be found using a simple program. A MATLAB function, \( \text{inverse}_{\text{inv}}(y) \), is given on the disk with this book for doing this. The result is

\[ \phi_m = 21.127^\circ \]

Next use Eq. (10.5) to compute

\[ r_m = \frac{r_{b_1}}{\cos \phi_m} = \frac{r_2 \cos \phi}{\cos \phi_m} = 4.1975 \]

and

\[ r_m = \frac{r_{b_1}}{\cos \phi_m} = \frac{r_2 \cos \phi}{\cos \phi_m} = \frac{N_3 \cos \phi}{2P_d \cos \phi_m} = \frac{13 \cos 20^\circ}{2 \times 6 \times \cos 21.127^\circ} = 1.0914 \text{ in} \]

The meshing center distance is

\[ C_m = 4.1975 + 1.0914 = 5.2889 \text{ in} \]

The standard center distance

\[ C_s = r_2 + r_3 = r_2 + \frac{N_3}{2P_d} = 4.1667 + \frac{13}{2 \times 6} = 5.2500 \text{ in} \]

### 10.12 Cartesian Coordinates of an Involute Tooth Generated with a Rack

It is often desirable to know the coordinates of the tooth profile for a given generating rack. This is of interest for drawing the gear or for machining the gear on a standard CNC milling machine without form cutters. If only the involute portion is required, it is relatively easy to compute the coordinates of points on the gear contact surface. However, if the entire profile is desired, the procedure is considerably more difficult. The problem can be approached directly if the geometry of the rack is known analytically. However, an indirect approach developed by Vijayakar is more general and applicable to a wide range of gear generators. This approach, which is relatively easy to program, is given here. The data required to generate the gear are a description of the rack, the number of teeth on the gear, and the outer diameter of the gear.

#### 10.12.1 Coordinate Systems

Figure 10.31 shows a coordinate system attached to the rack that is assumed to be fixed. The origin of the system is at the intersection of the centerline of a rack tooth and the pitch line of the rack. The coordinates of any point \( P \) on the surface of the rack can be defined by \((x_r, y_r)\) relative to the coordinate system fixed to the rack. Also, the outer normal to the rack